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1982 J. Phys. A: Math. Gen. 15 2865

(http://iopscience.iop.org/0305-4470/15/9/035)

J. Phys. A: Math. Gen. 15 (1982) 2865–2877. Printed in Great Britain

Anisotropic square lattice Potts ferromagnet: renormalisation group treatment

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Received 26 November 1981, in final form 28 April 1982

Abstract. The choice of a convenient self-dual cell within a real space renormalisation group framework enables a satisfactory treatment of the anisotropic square lattice q-state Potts ferromagnet criticality. The *exact* critical frontier and dimensionality crossover exponent ϕ as well as the expected universality behaviour (renormalisation flow sense) are recovered for any linear scaling factor b and all values of $q (q \leq 4)$. The $b = 2$ and $b = 3$ approximate correlation length critical exponent ν is calculated for all values of q and compared with the den Nijs conjecture. The same calculation is performed, for all values of b, for the exponent $\nu(d = 1)$ associated with the one-dimensional limit and the *exact* result $\nu(d = 1) = 1$ is recovered in the limit $b \to \infty$.

1. Introduction

During recent years considerable effort has been dedicated to the construction of real space renormalisation group (RG) frameworks suitable for the treatment of several models like the site and bond percolation, Ising and q-state Potts ones. A particular case which has frequently been focused upon is the anisotropic square lattice q-state Potts ferromagnet whose Hamiltonian is given by

$$\mathcal{H} = -q \sum_{\langle i,j \rangle} J_{ij} \delta_{\sigma_i \sigma_j} \qquad \sigma_i = 1, 2, \ldots, q \ \forall i \tag{1}$$

where $J_{ij} = J_x \geq 0$ ($J_{ij} = J_y \geq 0$) if sites i and j are 'horizontal' ('vertical') first neighbours (as a matter of fact, the present paper remains practically unchanged in the case where one or both coupling constants are negative). Any satisfactory RG proposal for this problem should recover the following facts.

(i) The transition is continuous (first order) if $0 \leq q \leq 4$ ($q > 4$) according to Baxter (1973), Straley and Fisher (1973), and Kim and Joseph (1975).

(ii) All properties of the system are invariant through $x \leftrightarrow y$ permutation.

(iii) The anisotropic square lattice is self-dual, therefore the dual transformation (Kim and Joseph 1975, Burkhardt and Southern 1978, Baxter *et al* 1978)

$$\exp(-qJ_x/k_B T) \leftrightarrow \frac{1 - \exp(-qJ_y/k_B T)}{1 + (q-1)\exp(-qJ_y/k_B T)} \tag{2}$$

interchanges its para- and ferromagnetic phases, and consequently the critical frontier is given by

$$t_x = t_y^{\mathrm{D}} \equiv \frac{1-t_y}{1+(q-1)t_y} \tag{3}$$

where we have introduced convenient variables (hereafter referred to as *transmissivities* (see Tsallis 1981, Tsallis and Levy 1981, and references therein), through

$$t_r \equiv \frac{1-\exp(-qJ_r/k_{\mathrm{B}}T)}{1+(q-1)\exp(-qJ_r/k_{\mathrm{B}}T)} \qquad r = x, y. \tag{4}$$

(iv) The system is *universal*, i.e. its critical behaviour for fixed q is one and the same for all non-vanishing values of J_x and J_y (in particular, the correlation length critical exponent ν is the same along the critical frontier excepted both one-dimensional limits $J_x = 0$ or $J_y = 0$).

(v) The crossover exponent ϕ associated with the one-dimensional limits equals one; this fact means that if we consider, for instance, the limit $J_y/J_x \to 0$, the critical frontier satisfies $t_y \propto 1-t_x$. It is clear that this weak restriction is satisfied by equation (3) which implies $t_y \sim (1-t_x)/q$.

(vi) The correlation length critical exponent $\nu(d=1)$ associated with the one-dimensional limits equals one.

(vii) The q dependence of the critical exponent ν (for $J_x, J_y \neq 0$) has not yet been rigorously established, however, the den Nijs (1979) conjecture, namely

$$\nu = \frac{2}{3[2+\pi/(\cos^{-1}\sqrt{q}/2-\pi)]} \tag{5}$$

$$\sim \pi/3\sqrt{q} \qquad \text{for } q \to 0 \tag{5'}$$

is possibly exact.

An RG treatment of the present problem consists in the construction of a two-dimensional recursive relation (generated by the renormalisation of an appropriate cell into a smaller one) which we shall denote

$$t'_x = R_b^x(t_x, t_y) \qquad t'_y = R_b^y(t_x, t_y) \tag{6}$$

where $b > 1$ is the linear scaling factor. This recursive relation is expected to provide fixed points (t_x^*, t_y^*) which satisfy

$$t_x^* = R_b^x(t_x^*, t_y^*) \qquad t_y^* = R_b^y(t_x^*, t_y^*) \tag{7}$$

as well as a Jacobian matrix

$$\begin{pmatrix} \partial t'_x/\partial t_x & \partial t'_x/\partial t_y \\ \partial t'_y/\partial t_x & \partial t'_y/\partial t_y \end{pmatrix} \tag{8}$$

whose eigenvalues and eigenvectors at each one of those fixed points are associated with relevant critical quantities. Let us note that it is by no means necessary (or even eventually convenient) to perform the renormalisation in a two-dimensional space ($t_x - t_y$ space in our case) and wider spaces can be used.

Let us now translate the restrictions (i)–(vii) into RG language.

(i′) An anomaly must appear, at $q = 4$, in the topology of the flow diagram while q varies; by anomaly we refer for instance to a bifurcation, terminal or turning point in the path of the relevant fixed points. However, it is not obvious that such anomaly can be observed without an enlargement of the renormalisation space (see, for example, Nienhuis *et al* 1979, Riedel 1981).

(ii′) It must be

$$R_b^y(y, x) = R_b^x(x, y) \equiv R_b(x, y). \tag{9}$$

This restriction leads to the invariance of the flow diagram through $t_x \leftrightarrow t_y$ permutation, i.e. there is a mirror symmetry with respect to the isotropic $t_x = t_y$ axis. The most satisfactory way of obtaining relation (9) is to use cells which themselves preserve the equivalence between the 'horizontal' and 'vertical' directions.

(iii′) It must be

$$R_b^x(x, y) = [R_b^y(y^D, x^D)]^D \equiv \frac{1 - R_b^y(y^D, x^D)}{1 + (q-1)R_b^y(y^D, x^D)} \tag{10}$$

where the superscript D denotes transformation (3) (see also Tsallis 1981, Tsallis and Levy 1981). The most satisfactory way for obtaining relation (10) is to use self-dual cells (a cell is said to be self-dual if it can be superimposed to itself in such a way that each one of its bonds is cut by one, and only one, bond of the original cell). The exact critical frontier (equation (3)) must be recovered as a flow line which runs between the one-dimensional limit points.

(iv′) A semi-stable fixed point must exist on the critical line in between the two one-dimensional limits, i.e. the eigenvalue (of the Jacobian matrix (8)), denoted λ_2, associated with the eigenvector tangential to the critical line must be less than one (the other eigenvalue, denoted λ_1, clearly must be greater than one).

(v′) At both one-dimensional limits, unstable fixed points must exist, and the associated Jacobian matrix must be proportional to unity ($\lambda_x = \lambda_y \equiv \lambda$), at least in the limit $b \to \infty$.

(vi′) The eigenvalue λ must be proportional to b in the limit $b \to \infty$ (we recall that $\nu(d=1) = \lim_{b\to\infty}(\ln b/\ln \lambda)$).

(vii′) The eigenvalue λ_1 must be such that $\nu = \lim_{b\to\infty}(\ln b/\ln \lambda_1)$ agrees with the possibly exact result (equation (5)).

Let us now place in the preceding context the recent RG literature on the subject. To the best of our knowledge, the unique RG treatment of the anisotropic q-state Potts model which is available is that performed by Kadanoff (1976). Within this approach only restrictions (ii′) and (iii′) are satisfied. In what concerns the isotropic model ($t_x = t_y$), only restrictions (i′), (iii′) and (vii′) are to be considered. Nienhuis *et al* (1979) qualitatively (but not quantitatively) satisfy these three restrictions. Blöte *et al* (1981) do not satisfy (i′) nor calculate the critical point (restriction (iii′)), but obtain, for $q<4$, a quite precise numerical approximation for ν (restriction (vii′)). Tsallis and Levy (1981) do not satisfy (i′), but obtain the exact critical point ($t_c = 1/(1+\sqrt{q})$), and acceptable numerical approximations for ν ($q<4$).

In what concerns the anisotropic system, some effort has been dedicated to the bond percolation problem (which corresponds to the particular case $q \to 1$, according to Kasteleyn and Fortuin (1969)). In this case, restriction (i′) need not be considered. In what concerns restrictions (ii′)–(vii′), Ikeda (1979) satisfies none of them, and Chaves *et al* (1979) and de Magalhães *et al* (1981) only satisfy (ii′) and (iii′), and obtain acceptable numerical approximations for ν (restriction (vii′)). Nakanishi *et al* (1981) only satisfy (ii′), (iv′), (v′) and (vi′); it must, however, be pointed out that they satisfy restriction (ii′) through an *ad hoc* procedure and not by considering a *single* cell whose 'horizontal' and 'vertical' spannings determine the corresponding recursive relations (equation (6)). Oliveira (1982) uses a suitable family of cells (Riera *et al* 1980, de Magalhães *et al* 1981, Curado *et al* 1981, Oliveira 1981, see figure 1) and simultaneously satisfies restrictions (ii′)–(vi′); the exact critical frontier $t_x + t_y = 1$ is obtained because, besides the fact that restrictions (ii′) and (iii′) are satisfied, each cell of this family reduces to a *single* linear chain in the one-dimensional limits (this important property is not satisfied by the cells used by Chaves *et al* (1979) and de Magalhães *et al* (1981); at the terminals of these cells different linear chains are being mixed).

In the present paper we follow along the lines of Oliveira (1982) and, by formulating the problem in terms of the transmissivities already mentioned, extend the RG treatment to the Potts model. By doing so, we satisfy restrictions (ii′)–(vi′) for all q and obtain a qualitatively acceptable q dependence of ν (restriction (vii′)); we fail, however, in what concerns restriction (i′).

2. Real space renormalisation group treatment

We shall use the family of self-dual cells indicated in figure 1. By using the break-collapse method (BCM; Tsallis and Levy 1981) we calculate the recursive relation (equation (6)) which renormalises the $b = 2$ cell (figure 1(*c*)) into the $b = 1$ cell (figure 1(*a*)) (we remark that a *single* pair of cells provides *both* t_x- and t_y-recurrences: it is enough to appropriately choose the input and output points, as illustrated, for $b = 1$, in figures 1(*a*) and (*b*)) and obtain

$$t'_x = R_2(t_x, t_y) \qquad t'_y = R_2(t_y, t_x) \tag{11}$$

with

$$\begin{aligned} R_2(t_x, t_y) \equiv [&t_x^3 + 4t_x^2 t_y + 3t_x t_y^2 + 2(q-2)t_x^3 t_y + 4(q-2)t_x^2 t_y^2 + 2(q-2)t_x^4 t_y \\ &+ (q^2+2q-5)t_x^3 t_y^2 + (4q-6)t_x^2 t_y^3 + (4q^2-13q+10)t_x^4 t_y^2 \\ &+ (6q^2-18q+12)t_x^3 t_y^3 + (q-2)t_x^2 t_y^4 + (q^2-5q+6)t_x^5 t_y^2 \\ &+ (2q^3-6q^2+10)t_x^4 t_y^3 + (3q^2-13q+14)t_x^3 t_y^4 \\ &+ (2q^3-12q^2+26q-20)t_x^5 t_y^3 + (3q^3-18q^2+38q-28)t_x^4 t_y^4 \\ &+ (q^4-7q^3+21q^2-30q+17)t_x^5 t_y^4]/[1 + 2(q-1)t_x t_y \\ &+ 2(q-1)t_x^3 t_y + (q^2-1)t_x^2 t_y^2 + (2q^2-6q+4)t_x^3 t_y^2 \\ &+ (2q^2-3q+1)t_x^4 t_y^2 + 2q(q-1)t_x^3 t_y^3 + (q-1)t_x^2 t_y^4 + (q^2-3q+2)t_x^5 t_y^2 \\ &+ (2q^3-4q^2-2q+4)t_x^4 t_y^3 + (3q^2-9q+6)t_x^3 t_y^4 \\ &+ (2q^3-10q^2+16q-8)t_x^5 t_y^3 + (3q^3-15q^2+24q-12)t_x^4 t_y^4 \\ &+ (q^4-7q^3+18q^2-20q+8)t_x^5 t_y^4]. \end{aligned} \tag{12}$$

This recursive relation (which, for $q = 1$, recovers that of Oliveira (1982)) presents two trivial stable fixed points (namely $(t_x^*, t_y^*) = (0, 0)$ and $(t_x^*, t_y^*) = (1, 1)$), two one-dimensional unstable fixed points (namely (1, 0) and (0, 1)) and one isotropic semi-stable fixed point (namely (t_c, t_c) with $t_c \equiv (\sqrt{q}+1)^{-1}$ which is the exact value), see figure 2. As a matter of fact the same set of fixed points will be obtained for all values of b.

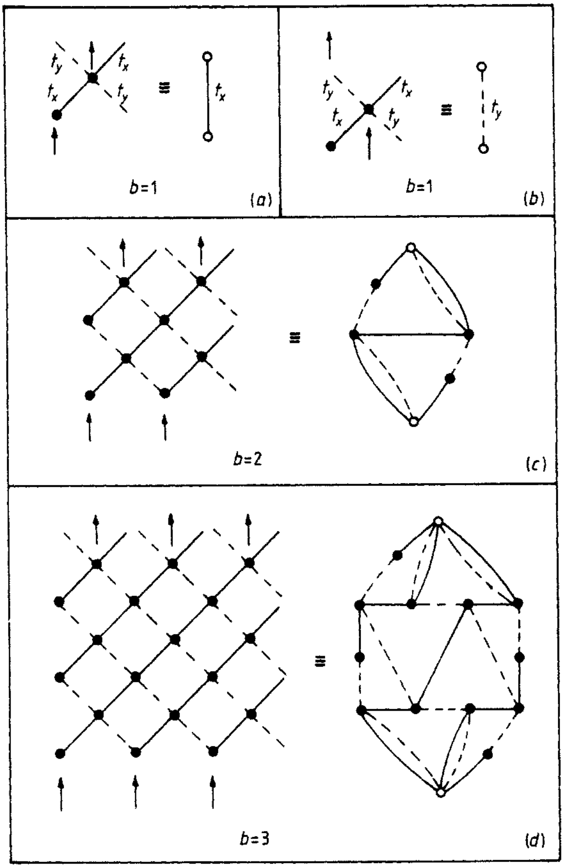

Figure 1. Self-dual cells and their two-rooted graph representation; all the entrances and all the exits of the cell, indicated by arrows, are to be respectively collapsed in order to generate the two roots or terminal sites (○) of the associated graph (see also Oliveira *et al* 1980); the internal sites of the cell become, without any modification, the internal sites (●) of the graph. These graphs provide $R_1(t_x, t_y) = t_x(a)$, $R_1(t_y, t_x) = t_y(b)$, $R_2(t_x, t_y)$ (c) and $R_3(t_x, t_y)$ (d) (we recall that the summation is carried out only over the internal sites of the graph).

Let us first analyse the isotropic fixed point. The Jacobian matrix (8) associated with equations (11) and (12) presents an eigenvalue (bigger than unity for any finite q)

$$\lambda_1(b=2) = (2025 + 11\,160\sqrt{q} + 26\,580q + 35\,792q^{3/2} + 29\,852q^2 + 15\,816q^{5/2} + 5207q^3 + 976q^{7/2} + 80q^4)/(2025 + 8820\sqrt{q} + 16\,804q + 18\,290q^{3/2} + 12\,444q^2 + 5424q^{5/2} + 1481q^3 + 232q^{7/2} + 16q^4) \tag{13}$$

associated with the eigenvector $(1, 1)/\sqrt{2}$ (which, in fact, will be the same for all values of b), and an eigenvalue (less than unity for any finite q)

$$\lambda_2(b=2) = (10\,125 + 88\,650\sqrt{q} + 342\,860q + 781\,853q^{3/2} + 1\,178\,008q^2 + 1\,240\,724q^{5/2} + 939\,667q^3 + 516\,906q^{7/2} + 205\,408q^4$$

$$+57\,611q^{9/2}+10\,844q^5+1232q^{11/2}+64q^6)/(91\,125+595\,350\sqrt{q}$$
$$+1\,782\,540q+3\,234\,167q^{3/2}+3\,960\,600q^2+3\,449\,388q^{5/2}$$
$$+2\,191\,343q^3+1\,023\,534q^{7/2}+349\,008q^4+84\,773q^{9/2}$$
$$+13\,932q^5+1392q^{11/2}+64q^6) \quad (14)$$

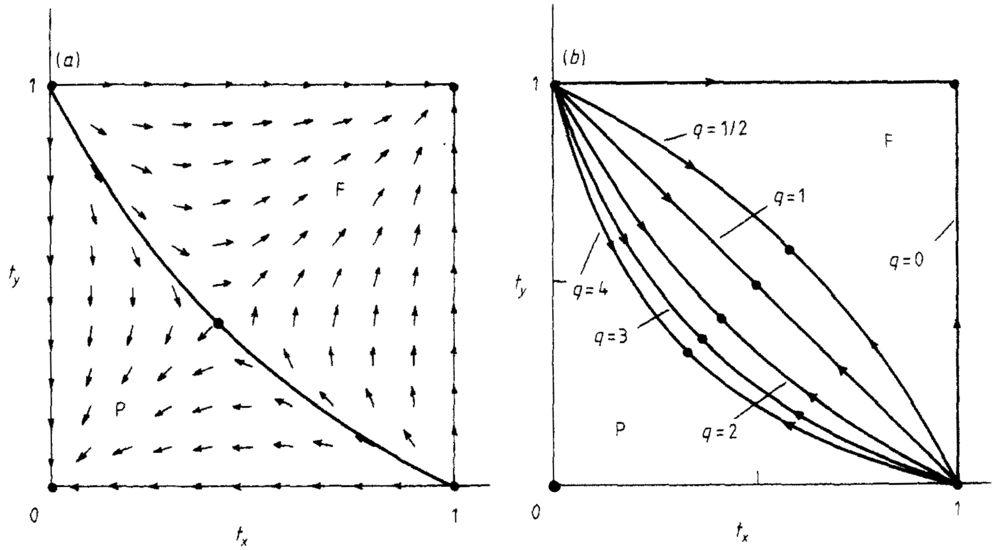

Figure 2. Flow diagrams associated with equation (11). The dots (full curve) represent(s) fixed points (the critical flow line; it coincides with the *exact* result $t_x = t_y^D$ (equation (3))). (P)((F)) denotes the paramagnetic (ferromagnetic) phase. (*a*) Complete $b = 2$ flow diagram for $q = 2$; (*b*) critical flow lines associated with various values of q and any value of b (the limit $q \to 0$ corresponds to tree-like percolation; the q- and q^{-1}-frontiers are, for all values of q, symmetric with respect to the straight line $t_x + t_y = 1$).

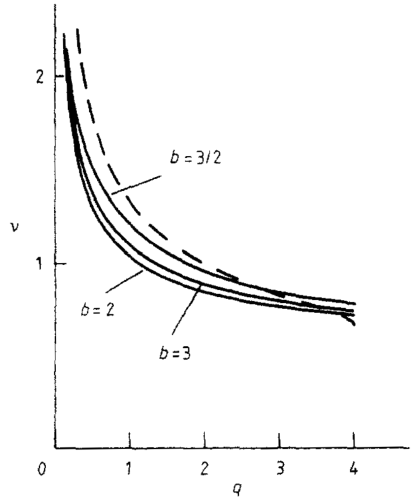

Figure 3. q dependence of the correlation length critical exponent ν; the full (broken) curves correspond to the present RG results (to the den Nijs (1979) conjecture). By $b = \frac{3}{2}$ we mean the value obtained by renormalising the $b = 3$ cell into the $b = 2$ one, hence $\nu(b = \frac{3}{2}) = (\ln \frac{3}{2})/\ln[(\lambda_1(b = 3))/(\lambda_1(b = 2))]$.

Table 1. RG and conjectural values of the critical exponent ν. (a) See caption of figure 3; (b) these values coincide with those appearing in Oliveira (1982).

	$q\to 0$	$q=1$	$q=2$	$q=3$	$q=4$
$b=2$	$\frac{45 \ln 2}{52}\frac{1}{\sqrt{q}}$ $\simeq 0.600/\sqrt{q}$	$\frac{\ln 2}{\ln \frac{249}{2^7}}$ $\simeq 1.042$ (b)	$\frac{\ln 2}{\ln \frac{29}{13}}$ $\simeq 0.864$	$\frac{\ln 2}{\ln \frac{82\,917+47\,872\sqrt{3}}{34\,286+19\,795\sqrt{3}}}$ $\simeq 0.785$	$\frac{\ln 2}{\ln \frac{2193}{857}}$ $\simeq 0.738$
$b=3$	$\frac{3625 \ln 3}{5996}\frac{1}{\sqrt{q}}$ $\simeq 0.664/\sqrt{q}$	$\frac{\ln 3}{\ln \frac{5\,700\,575}{2^{21}}}$ $\simeq 1.099$ (b)	$\frac{\ln 3}{\ln 3.3921}$ $\simeq 0.899$	$\frac{\ln 3}{\ln 3.8777}$ $\simeq 0.811$	$\frac{\ln 3}{\ln 4.2643}$ $\simeq 0.758$
$b=3/2$ (a)	$\frac{32\,625 \ln(3/2)}{16\,264}\frac{1}{\sqrt{q}}$ $\simeq 0.813/\sqrt{q}$	$\frac{\ln(3/2)}{\ln \frac{5\,700\,575}{249\times 2^{14}}}$ $\simeq 1.212$	$\frac{\ln(3/2)}{\ln 1.5206}$ $\simeq 0.967$	$\frac{\ln(3/2)}{\ln 1.6034}$ $\simeq 0.859$	$\frac{\ln(3/2)}{\ln 1.6664}$ $\simeq 0.794$
Conjecture (den Nijs 1979)	$\frac{\pi}{3}\frac{1}{\sqrt{q}} \simeq 1.047/\sqrt{q}$	$\frac{4}{3} \simeq 1.333$	1	$\frac{5}{6} \simeq 0.833$	$\frac{2}{3} \simeq 0.667$

Table 2. Coefficients of the numerator ($\{n_j^{(t)}\}$; top value) and denominator ($\{d_j^{(t)}\}$; bottom value) of $R_3(t, t)$ (equation (16)); all missing coefficients vanish.

	q^{12}	q^{11}	q^{10}	q^9	q^8	q^7	q^6	q^5	q^4	q^3	q^2	q^1	q^0
t^4											0	0	0
											6	4	−10
t^5											0	0	52
											10	−30	20
t^6										0	0	46	−92
										6	22	−29	1
t^7										0	10	184	−217
										22	−35	−49	62
t^8									0	0	228	−480	48
									5	46	−57	36	−30
t^9									0	70	319	−1 102	893
									20	118	−558	548	−128
t^{10}								0	6	515	−1 064	−525	1 090
								2	121	−386	721	−469	11
t^{11}								0	183	723	−4 258	4 617	23
								26	78	136	−1 297	963	94
t^{12}							0	24	944	−2 276	−4 238	14 970	−10 652
							1	84	439	−2 342	3 204	−1 316	−70
t^{13}							1	322	1 545	−13 171	23 993	−9 728	−5 082
							12	402	−879	−295	−1 548	6 676	−4 368
t^{14}							44	1 684	−6 948	−2 235	43 626	−64 269	26 378
							107	698	−2 763	−4 616	24 486	−27 143	9 231

t^{15}						2	565	691	−18 558	42 975	8 663	−118 459	97 735
						8	495	273	−12 816	30 030	−9 185	−31 079	22 274
t^{16}						85	1 720	−8 275	−24 764	226 588	−562 754	620 598	−262 820
						95	1 545	−8 963	−2 419	94 854	−221 797	206 676	−69 991
t^{17}					5	613	1 317	−47 733	238 848	−541 629	602 256	−264 552	−4 770
					5	661	−387	−28 119	134 983	−260 803	222 492	−55 752	−13 080
t^{18}					96	2 049	−23 010	73 686	−9 690	−502 699	1 333 460	−1 483 919	638 470
					106	1 643	−17 981	48 660	27 403	−385 954	790 677	−692 555	228 001
t^{19}				6	670	−3 482	−21 155	253 992	−1 059 645	2 425 249	−3 247 444	2 397 699	−755 935
				6	646	−3 136	−20 232	216 118	−803 290	1 592 252	−1 785 127	1 064 401	−261 638
t^{20}				100	822	−21 875	156 168	−593 712	1 369 776	−1 949 916	1 606 006	−616 482	34 252
				100	808	−20 885	141 030	−490 388	989 517	−1 153 738	686 392	−113 320	−39 516
t^{21}			6	426	−6 242	35 850	−100 740	86 250	344 761	−1 365 061	2 258 237	−1 923 936	691 514
			6	426	−6 162	34 040	−85 830	29 262	431 471	−1 288 621	1 769 922	−1 236 910	352 396
t^{22}			68	−616	−292	31 567	−223 782	862 934	−2 145 912	3 570 355	−3 882 278	2 513 723	−738 486
			68	−616	−342	31 949	−221 094	818 160	−1 906 617	2 890 198	−2 773 986	1 530 688	−368 408
t^{23}		4	56	−1 798	17 902	−102 164	387 934	−1 038 883	1 997 700	−2 727 555	2 528 469	−1 433 673	376 314
		4	56	−1 798	17 862	−100 988	375 229	−966 137	1 746 876	−2 184 472	1 800 428	−878 332	191 272
t^{24}		17	−340	3 204	−18 802	76 462	−226 564	499 499	−819 252	978 516	−808 826	415 122	−99 836
		17	−340	3 204	−18 748	75 486	−218 840	464 259	−717 462	787 596	−581 732	258 640	−52 080
t^{25}	1	−21	210	−1 322	5 841	−19 107	47 502	−90 479	131 006	−140 449	105 595	−49 834	11 121
	1	−21	210	−1 322	5 825	−18 873	45 945	−84 337	115 333	−114 031	76 962	−31 652	5 960

associated with the eigenvector $(-1, 1)/\sqrt{2}$ (the same for all values of b). The fact that $\lambda_2 < 1$ enables restriction (iv′) to be satisfied. The q dependence of the approximate critical exponent $\nu(b=2) = \ln 2/\ln \lambda_1(b=2)$ is presented in figure 3 and table 1. It is clear that $\lambda_1(b=2)$ could have been obtained directly from the isotropic case $(t_x = t_y \equiv t)$ whose recursive relation is given by

$$\begin{aligned} t' = R_2(t,t) = [&8t^3 + 6(q-2)t^4 + (q^2+8q-15)t^5 + (10q^2-30q+20)t^6 \\ &+ (2q^3-2q^2-18q+30)t^7 + (5q^3-30q^2+64q-48)t^8 + (q^4-7q^3+21q^2 \\ &-30q+17)t^9]/[1+2(q-1)t^2+(q^2+2q-3)t^4+(2q^2-6q+4)t^5 \\ &+4q(q-1)t^6+(2q^3-14q+12)t^7+(5q^3-25q^2+40q-20)t^8 \\ &+(q^4-7q^3+18q^2-20q+8)t^9] \end{aligned} \tag{15}$$

hence

$$\nu(b=2) = \ln 2/\ln(\mathrm{d}R_2(t,t)/\mathrm{d}t)_{t=1/(\sqrt{q}+1)} = \ln 2/\ln \lambda_1(b=2).$$

For $b = 3$ we have calculated (by using the BCM) the isotropic case and have obtained

$$t' = R_3(t,t) = \frac{\sum_{i=5}^{25} \left(\sum_{j=0}^{12} n_j^{(i)} q^{12-j}\right) t^i}{1 + 4(q-1)t^2 + \sum_{i=4}^{25} \left(\sum_{j=0}^{12} d_j^{(i)} q^{12-j}\right) t^i} \tag{16}$$

where the integer coefficients $\{n_j^{(i)}\}$ and $\{d_j^{(i)}\}$ are presented in table 2. From this expression we straightforwardly obtain

$$\lambda_1(b=3) = \left.\frac{\mathrm{d}R_3(t,t)}{\mathrm{d}t}\right|_{t=1/(\sqrt{q}+1)} = \frac{\sum_{j=0}^{24} \alpha_j q^{j/2}}{\sum_{j=0}^{24} \beta_j q^{j/2}} \tag{17}$$

where the coefficients $\{\alpha_j\}$ and $\{\beta_j\}$ are presented in table 3. The associated critical exponent $\nu(b=3) = \ln 3/\ln \lambda_1(b=3)$ is presented in figure 3 and table 1.

Let us now turn our attention onto the one-dimensional fixed points. The Jacobian matrix (8) associated with equations (11) and (12) is degenerate (i.e. proportional to the unity matrix) therefore the dimensionality crossover exponent ϕ equals one, which is the exact result. The degenerate eigenvalue is $\lambda(b=2) = 3$ (bigger than unity as

Table 3. Coefficients of the numerator ($\{\alpha_j\}$) and denominator ($\{\beta_j\}$) of $\lambda_1(b=3)$ (equation (17)).

j	α_j	β_j	j	α_j	β_j
0	26 609 765 625	26 609 765 625	12	37 326 398 614 887	8 730 873 263 387
1	345 165 975 000	301 151 587 500	13	18 238 527 887 576	3 830 174 165 004
2	2 136 160 842 300	1 632 004 524 900	14	7 595 924 956 680	1 434 397 590 136
3	8 398 260 105 840	5 637 003 575 850	15	2 689 378 468 288	457 379 903 606
4	23 558 927 138 490	13 935 002 902 530	16	805 456 074 350	123 565 884 406
5	50 208 537 095 364	26 244 158 374 710	17	202 466 608 404	28 065 470 706
6	84 507 606 761 853	39 134 969 393 899	18	42 226 821 982	5 298 478 858
7	115 275 290 061 296	47 406 621 773 750	19	7 186 504 632	817 846 452
8	129 749 686 604 187	47 487 604 806 763	20	973 987 016	100 745 336
9	122 049 367 890 464	39 833 254 423 768	21	101 289 672	9 544 940
10	96 809 133 215 685	28 227 060 346 311	22	7 605 120	654 600
11	65 144 830 142 464	16 998 851 657 994	23	367 800	29 000
			24	8 625	625

expected). As a matter of fact, for any value of b, the recursive relation in the vicinity of a one-dimensional fixed point (let us say $(t_x^*, t_y^*) = (1, 0)$) leads to an eigenvalue $\lambda(b)$ which is that of a linear chain (along the x direction in our case). The recurrence is given by

$$t_x' = R_b(t_x, 0) = t_x^{2b-1} \tag{18}$$

hence

$$\lambda(b) \equiv \left.\frac{\mathrm{d}R_b(t_x, 0)}{\mathrm{d}t_x}\right|_{t_x=1} = 2b - 1$$

and finally

$$\nu(d=1) = \lim_{b\to\infty} \frac{\ln b}{\ln \lambda(b)} = \lim_{b\to\infty} \frac{\ln b}{\ln(2b-1)} = 1 \tag{19}$$

which is the exact result.

3. The s variable

In order to make a remark let us introduce a new variable (Tsallis 1981, Tsallis and de Magalhães 1981) namely

$$s_r \equiv s(t_r) \equiv \frac{\ln[1+(q-1)t_r]}{\ln q} \qquad r = x, y. \tag{20}$$

It is straightforward through the use of

$$t_r^{\mathrm{D}} \equiv \frac{1-t_r}{1+(q-1)t_r} \qquad r = x, y \tag{21}$$

to verify that

$$s^{\mathrm{D}}(t_r) \equiv s(t_r^{\mathrm{D}}) = 1 - s(t_r) \qquad r = x, y \tag{22}$$

and that the critical frontier (3) can be rewritten in a *universal* form (the same for all values of q) namely

$$s_x + s_y = 1 \tag{23}$$

which is precisely that of bond percolation ($q \to 1$). Consequently we can define the RG in an alternative manner, namely

$$\begin{aligned} s_x' &\equiv s(t_x') = s(R_b(t_x, t_y)) = s\left(R_b\left(\frac{q^{s_x}-1}{q-1}, \frac{q^{s_y}-1}{q-1}\right)\right) \\ s_y' &\equiv s(t_y') = s(R_b(t_y, t_x)) = s\left(R_b\left(\frac{q^{s_y}-1}{q-1}, \frac{q^{s_x}-1}{q-1}\right)\right). \end{aligned} \tag{24}$$

The flow diagram presents, for *all values of* q, one and the same set of fixed points (namely $(s_x^*, s_y^*) = (0, 0)$, $(1, 1)$, $(1, 0)$, $(0, 1)$ and $(\frac{1}{2}, \frac{1}{2})$) and critical flow line (namely that of equation (23)), i.e. it presents the RG topology of the bond percolation problem.

In what concerns the critical exponents nothing is changed with respect to the RG in the t variables as, for any fixed point, we have

$$\begin{pmatrix} \partial s'_x/\partial s_x & \partial s'_x/\partial s_y \\ \partial s'_y/\partial s_x & \partial s'_y/\partial s_y \end{pmatrix} = \begin{pmatrix} \partial t'_x/\partial t_x & \partial t'_x/\partial t_y \\ \partial t'_y/\partial t_x & \partial t'_y/\partial t_y \end{pmatrix}. \tag{25}$$

4. Conclusion

The use of appropriate cells (which are *self-dual* and in the one-dimensional limits reduce to *single* chains) enables us to reproduce, within a simple real space renormalisation group, a considerable quantity of *exact* results (points (ii)–(vi) of § 1) concerning the criticality of the anisotropic square lattice q-state Potts model. In what concerns the q dependence of the correlation length critical exponent ν (point (vii) of § 1) we obtain results which are compatible with the den Nijs (1979) conjecture and which improve with increasing cell size as long as q is not too close to 4; on the whole they are quite similar to those obtained by Tsallis and Levy (1981) and reinforce the den Nijs (1979) conjecture in the limit $q \to 0$ (tree-like percolation) as they all provide $\nu \propto 1/\sqrt{q}$. In what concerns point (i) of § 1 we have failed, i.e. nothing special occurs at $q = 4$ (nor at any other finite value of q); the fact that we have not enlarged the parameter space (our renormalisation is restricted to the (t_x, t_y) space) is, according to the ideas contained in Nienhuis *et al* (1979), quite probably at the origin of this failure.

It is interesting to compare the present results with those of Kadanoff (1976) for the same system. Kadanoff discusses the 'troubles with the approximation' he introduces, namely: (1) the $d = 2$ to $d = 1$ crossover is completely missed; (2) a considerable inaccuracy in the determination of the value for ν is found; (3) the procedure leads to $\beta/\nu = 0$ (which, for $d = 2$, implies that the spin–spin correlation function critical exponent η vanishes). Difficulties (1) and (2) are absent from the present treatment; difficulty (3) is out of the scope of this work as we calculate neither a magnetic-type critical exponent nor quantities directly related to it (let us say, however, that this difficulty can be avoided by using procedures like those recently introduced by Martin and Tsallis (1981a, b); these procedures extend the present one which is recovered as one of its stages).

Incidentally we present (in § 3) a renormalisation group (constructed in the (s_x, s_y) space instead of the (t_x, t_y) one) which has interesting universal properties: the set of fixed points and critical flow line (critical frontier) is independent of q and is that of bond percolation.

Acknowledgment

We acknowledge with pleasure computational assistance from G Schwachheim and E M F Curado.

References

Baxter R J 1973 *J. Phys. C: Solid State Phys.* **6** L445
Baxter R J, Temperley H N V and Ashley S E 1978 *Proc. R. Soc.* A **358** 535

Blöte H W, Nightingale M P and Derrida B 1981 *J. Phys. A: Math. Gen.* **14** L45
Burkhardt T W and Southern B W 1978 *J. Phys. A: Math. Gen.* **11** L247
Chaves C M, Oliveira P M C, Riera R and de Queiroz S L A 1979 *Prog. Theor. Phys.* **62** 1550
Curado E M F, Tsallis C and Schwachheim G 1981 to be published
Ikeda H 1979 *Prog. Theor. Phys.* **61** 842
Kadanoff L P 1976 *Ann. Phys.* **100** 359
Kasteleyn P W and Fortuin C M 1969 *J. Phys. Soc. Japan suppl.* **26** 11
Kim D and Joseph R I 1975 *J. Phys. A: Math. Gen.* **8** 891
de Magalhães A C N, Tsallis C and Schwachheim G 1981 *J. Phys. C: Solid State Phys.* **14** 1393
Martin H O and Tsallis C 1981a *J. Phys. C: Solid State Phys.* **14** 5645
—— 1981b *Z. Phys.* B **44** 325
Nakanishi H, Reynolds P J and Redner S 1981 *J. Phys. A: Math. Gen.* **14** 855
Nienhuis B, Berker A N, Riedel E K and Schick M 1979 *Phys. Rev. Lett.* **43** 737
den Nijs M P M 1979 *J. Phys. A: Math. Gen.* **12** 1857
Oliveira P M C 1981 *Phys. Rev. Lett.* **47** 1423
—— 1982 *Phys. Rev.* B **25** 2034
Oliveira P M C, de Queiroz S L A, Riera R and Chaves C M 1980 *J. Phys. A: Math. Gen.* **13** 2457
Riedel E K 1981 *Physica* A **106** 110
Riera R, Oliveira P M C, Chaves C M and de Queiroz S L A 1980 *Phys. Rev.* B **22** 3481
Straley J P and Fisher M E 1973 *J. Phys. A: Math., Nucl. Gen.* **6** 1310
Tsallis C 1981 *J. Phys. C: Solid State Phys.* **14** L85
Tsallis C and Levy S V F 1981 *Phys. Rev. Lett.* **47** 950
Tsallis C and de Magalhães A C N 1981 *J. Physique Lett.* **42** L227